Mid-Semestral Exam
Algebra-II
M. Math - First year
2013-2014

Time: 3 hrs
Max score: 100

Answer question 1 and any 5 from the rest.

(1) State true or false. Justify your answers. No marks will be awarded in the absence of proper justification.
(i) Every algebraic extension is a finite extension.
(ii) Every finite normal extension of a field F is a splitting field of a polynomial in $F[x]$.
(iii) Let $f(x)$ be a non-constant polynomial over a finite field F, such that derivative of $f(x)$ is zero. Then $f(x)$ is irreducible.
(iv) Let $F \subseteq L \subseteq K$ be fields, such that $L|F$ and $K|L$ are both Galois extensions. Then $K|F$ is Galois. 5+5+5+5

(2) (i) For any field F, show that there exists an algebraically closed field K containing F.
(ii)Let $f(x) \in \mathbb{Q}[x]$ be irreducible over $\mathbb{Q}$. Let F be the splitting field of $f(x)$ over $\mathbb{Q}$. If $[F : \mathbb{Q}]$ is odd, prove that all roots of $f(x)$ are real. 10+6

(3) Show that the cyclotomic extension $\mathbb{Q}(\zeta_n)|\mathbb{Q}$, where ζ_n is a primitive nth root of unity is of degree $\phi(n)$ over $\mathbb{Q}$ (ϕ denotes the Euler's phi-function). 16

(4) (a) Let $K|F$ be a finite Galois extension. Suppose that $a \in K$ satisfies $\sigma(a) \neq a$ for all $\sigma \in Gal(K|F)$, $\sigma \neq 1$. Prove that $F(a) = K$.
(b) Let ζ be a primitive 8th root of unity over $\mathbb{Q}$. Let p be an odd prime integer.
(i) Prove that $\sqrt{p} \notin \mathbb{Q}(\zeta)$.
(ii) Prove that $\mathbb{Q}(\zeta, \sqrt{p}) = \mathbb{Q}(\zeta + \sqrt{p})$. 6+10

(5) (a) State fundamental theorem of Galois theory.
(b) Suppose that $F = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = E$, where $E|F$ is a Galois extension, and that the intermediate field K_i corresponds to the subgroup H_i under the Galois correspondence. Show that $K_i|K_{i-1}$ is normal (hence Galois) if and only if H_i is a normal subgroup of H_{i-1}, and in this case, $\mathrm{Gal}(K_i/K_{i-1})$ is isomorphic to H_{i-1}/H_i. 6+10

(6) (a) Suppose K is a finite, separable, normal extension of a field F and L_1 and L_2 are normal extensions of F in K. Show the smallest field L in K containing L_1 and L_2 is a normal extension of F.

(b) Find a Galois extension K of $\mathbb{Q}$ such that $Gal(K|\mathbb{Q})$ is isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_8$. 6+10

(7) (a) Compute the splitting field K of the polynomial $f(x) = x^4 - 2$ over $\mathbb{Q}$.
(b) Show that K is a Galois extension of $\mathbb{Q}$.
(c) Compute the degree of the extension $[K : \mathbb{Q}]$.
(d) Identify the Galois group $Gal(K|\mathbb{Q})$.
(e) What are all the subgroups of $Gal(K|\mathbb{Q})$?
(f) What are all the intermediate subfields of $K|\mathbb{Q}$?
(g) Among the intermediate subfields, which are normal?
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